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An inequality for chromatic polynomials

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Abstract

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It is proved that if $P(G, t)$ is the chromatic polynomial of a simple graph G with n vertices, m edges, c components and b blocks, and if $t \leqslant 1$, then

$$|P(G, t)| \geqslant |t^c(t-1)^b|(1 + \gamma s + \gamma s^2 + \cdots + \gamma s^{\mu-1} + s^\mu),$$

where $\gamma = m - n + c$, $\mu = n - c - b$ and $s = 1 - t$. Equality holds for several classes of graphs with few circuits.

1. Introduction and motivation

Throughout this paper, G will denote a graph with n vertices, m edges, c components, b blocks and circuit rank $\gamma := m - n + c$, and μ will be defined by $\mu := n - c - b$. (Isolated vertices will count as components but not as blocks.) The corresponding numbers for a graph G_i will be denoted by n_i, m_i, c_i, b_i, γ_i and μ_i. Let $P(G, t)$ denote the chromatic polynomial of G. Parts (a)–(c) of the following theorem can be found in, for example, Tutte [2]; part (d) was proved by Woodall [4] and Whitehead and Zhao [3].

Theorem 1. *Let G be a simple graph.*

(a) *If $t < 0$, then $P(G, t)$ is nonzero with the sign of $(-1)^n$.*

(b) *At 0, $P(G, t)$ has a zero of multiplicity c (hence, a simple zero if G is connected).*

(c) *If $0 < t < 1$, then $P(G, t)$ is nonzero with the sign of $(-1)^{n-c}$.*

(d) *At 1, $P(G, t)$ has a zero of multiplicity b (hence, a simple zero if G is 2-connected).*

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This pattern cannot continue: it follows from the above results that if G is any 2-connected bipartite graph with an odd number of vertices, then $P(G, t)$ is negative just to the right of 1 and so has a zero between 1 and 2. The smallest example is $K_{2,3}$, which is also planar, However, let us define a *plane near-triangulation* to be a loopless multigraph G, necessarily 2-connected, drawn in the plane in such a way that one face is bounded by a circuit of $k \geqslant 3$ edges and every other face is bounded by a triangle: G is a *triangulation* if $k = 3$. For plane near-triangulations, the above pattern does continue as follows.

Theorem 2. *Let G be a plane near-triangulation, and let m' be the smallest number of edges whose deletion from G leaves a (simple) graph.*

(a) *If $1 < t < 2$, then $P(G, t)$ is nonzero with the sign of $(-1)^n$.*

(b) *At 2, $P(G, t)$ has a zero of multiplicity at least $m' + 1$, with equality if G is a triangulation. Thus $P(G, t)$ has a simple zero at 2 if G is a simple triangulation.*

Let us write $A \geqslant_x B$ if A and B can be expressed as polynomials in x and, when this is done, each coefficient in A is at least as large as the corresponding coefficient in B. Theorem 2, and all the consequences of Theorem 1 for near-triangulations, follow from the following result, which can easily be derived from the theorem on page 397 of Birkhoff and Lewis [1] (see also Theorem 5 in [4]).

Theorem 3. *Let G be a plane near-triangulation whose exceptional face has $k \geqslant 3$ edges. With m' as in Theorem 2, define $q(G, t)$ by*

$$P(G, t) = (-1)^{n-3-m'} t(t-1)(t-2)^{m'+1} q(G, t).$$

Then $q(G, t)$ is a polynomial in t and

$$q(G, t) \geqslant_r r^{k-3}(1+r)^{n-k-m'}$$

where $r := 2 - t$. Thus $q(G, t) \geqslant (2-t)^{k-3}(3-t)^{n-k-m'}$ if $t \leqslant 2$.

The present paper is devoted to a proof of the following theorem, which arose in an attempt to find a result in the spirit of Theorem 3 that would imply Theorem 1 for arbitrary graphs in a similar way.

Theorem 4. *Let G be a simple graph. Define $q(G, t)$ by*

$$P(G, t) = (-1)^{\mu} t^c (t-1)^b q(G, t).$$

Then $q(G, t)$ is a polynomial in t and

$$q(G, t) \geqslant_s 1 + \gamma s + \gamma s^2 + \cdots + \gamma s^{\mu-1} + s^{\mu} \tag{1}$$

where $s := 1 - t$. Thus $q(G, t) \geqslant 1$ if $t \leqslant 1$.

Note that $\gamma \geq 0$ and $\mu \geq 0$, with equality in each case if and only if G is a forest; and if $\mu = 1$ then G is circuit-free apart from a single triangle, so that $\gamma = 1$ also. It is easy to check that equality holds in (1) if G is a forest, or if the cycle space of G is spanned by a single circuit of length l (when $\gamma = 1$ and $\mu = l - 2$), or by a circuit of length l and a triangle ($\gamma = 2$, $\mu = l - 1$), or by three triangles not forming a K_4 ($\gamma = 3$, $\mu = 3$).

In proving Theorem 4 we shall need the deletion-contraction formula, which says that, for each edge e of a graph G,

$$P(G, t) = P(G - e, t) - P(G/e, t), \tag{2}$$

where $G - e$ and G/e are obtained from G by, respectively, deleting and contracting the edge e. We also need the well-known result that if $G = G_1 \cup G_2$ where $G_1 \cap G_2 = K_r$, then

$$P(G, t) = \frac{P(G_1, t)P(G_2, t)}{P(K_r, t)}, \tag{3}$$

and $P(K_r, t) = t(t-1) \cdots (t - r + 1)$. Since the chromatic polynomial is multiplicative over components, we can extend (3) to the case $r = 0$ by allowing the existence of the empty graph K_0 with $P(K_0, t) := 1$.

2. Proof of Theorem 4

We prove the result by induction on $m + n$. There are three cases to consider.

Case 1: $G = G_1 \cup G_2$ *where* $G_1 \cap G_2 = K_0$ *or* K_1 *or* K_2, $n_1 < n$ *and* $n_2 < n$.

(Recall that n_i denotes the number of vertices of G_i, etc.) The values of various parameters are as follows.

$G_1 \cap G_2$	$m_1 + m_2$	$n_1 + n_2$	$c_1 + c_2$	$b_1 + b_2$
K_0	m	n	c	b
K_1	m	$n + 1$	$c + 1$	b
K_2	$m + 1$	$n + 2$	$c + 1$	$b + 1$

Note that $\gamma_1 + \gamma_2 = \gamma$ and $\mu_1 + \mu_2 = \mu$ in each case. In view of the above values, we may write (3) in the form

$$P(G, t) = \frac{P(G_1, t)P(G_2, t)}{t^{c_1+c_2-c}(t-1)^{b_1+b_2-b}}$$

when $r = 0$, 1 or 2, from which it follows that

$$q(G, t) = q(G_1, t)q(G_2, t) \tag{4}$$

in each case. We may suppose inductively that the result holds for G_1 and G_2.

Thus we may deduce from (4) that $q(G, t)$ is a polynomial in t and

$$\begin{aligned} q(G, t) &\geqslant_s (1 + \gamma_1 s + \gamma_1 s^2 + \cdots + \gamma_1 s^{\mu_1 - 1} + s^{\mu_1}) \\ &\quad \times (1 + \gamma_2 s + \gamma_2 s^2 + \cdots + \gamma_2 s^{\mu_2 - 1} + s^{\mu_2}) \\ &\geqslant_s 1 + \gamma s + \gamma s^2 + \cdots + \gamma s^{\mu - 1} + s^{\mu} \end{aligned}$$

as required. (Recall from Section 1 that if $\mu_i = 0$ or 1, then $\gamma_i = \mu_i$.)

Case 2: *G is K_1, K_2 or K_3.*

Then we have the following values.

G	c	b	γ	μ	$P(G, t)$	$t^c(t-1)^b$	$q(G, t)$
K_1	1	0	0	0	t	t	1
K_2	1	1	0	0	$t(t-1)$	$t(t-1)$	1
K_3	1	1	1	1	$t(t-1)(t-2)$	$t(t-1)$	$1+(1-t)$

The result clearly holds.

Case 3: *Neither Case* 1 *nor Case* 2 *applies.*

Then G is connected with no cut-vertex, $|V(G)| \geqslant 4$, and G is not separated by any two adjacent vertices. Thus G is 2-connected, $G \neq K_3$, and if $e \in E(G)$ then G/e is 2-connected. Choose $e \in E(G)$, let $G_1 := G - e$, and let G_2 be the simple graph obtained from G/e by removing redundant multiple edges: clearly $P(G_2, t) = P(G/e, t)$. We may suppose inductively that the result holds for G_1 and G_2. Note that $m_1 = m - 1$, $m_2 \leqslant m - 1$ with equality iff G/e is simple, $n_1 = n$, $n_2 = n - 1$, $c_1 = c_2 = c = 1$, $b_2 = b = 1$, $\gamma_1 = \gamma - 1$, $\gamma_2 \leqslant \gamma$ with equality iff G/e is simple, $\mu_1 = \mu - b_1 + 1$ and $\mu_2 = \mu - 1$. Thus, by (2),

$$\begin{aligned} q(G, t) &= (-1)^{\mu_1 - \mu}(t-1)^{b_1 - 1} q(G_1, t) - (-1)^{\mu_2 - \mu} q(G_2, t) \\ &= (1-t)^{b_1 - 1} q(G_1, t) + q(G_2, t), \end{aligned} \tag{5}$$

which is a polynomial in t since $q(G_1, t)$ and $q(G_2, t)$ are.

There are now two subcases to consider.

Case 3a: *e can be chosen so that it does not lie in a triangle.*

Then G/e is simple, and

$$q(G_2, t) \geqslant_s 1 + \gamma s + \gamma s^2 + \cdots + \gamma s^{\mu - 2} + s^{\mu - 1} \tag{6}$$

by the induction hypothesis. The induction hypothesis also ensures that

$$\begin{aligned} (1-t)^{b_1 - 1} q(G_1, t) &\geqslant_s s^{\mu - \mu_1}[1 + (\gamma - 1)s + \cdots + (\gamma - 1)s^{\mu_1 - 1} + s^{\mu_1}] \\ &\geqslant_s (\gamma - 1)s^{\mu - 1} + s^{\mu} \end{aligned} \tag{7}$$

since $\gamma_1 = \gamma - 1 = 1$ if $\mu_1 = 1$ and $\gamma_1 = \gamma - 1 = 0$ if $\mu_1 = 0$. The result follows from (5), (6) and (7).

Case 3b: *Every edge of G lies in a triangle.*

Since $G \neq K_3$, it is not difficult to find an edge e such that $G_1 = G - e$ is 2-connected, so that $b_1 = 1$, $\mu_1 = \mu$ and (5) gives

$$q(G, t) = q(G_1, t) + q(G_2, t). \tag{8}$$

The induction hypothesis gives

$$q(G_1, t) \geqslant_s 1 + (\gamma - 1)s + \cdots + (\gamma - 1)s^{\mu-1} + s^{\mu} \tag{9}$$

and

$$q(G_2, t) \geqslant_s 1 + s + s^2 + \cdots + s^{\mu-2} + s^{\mu-1} \tag{10}$$

since $\gamma_2 \geqslant 1$ (because G_2 is 2-connected, and $\gamma_2 = 0$ would imply that G_2 is a forest). The result follows from (8), (9) and (10). □

References

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